7. Vorovich, I. I., On the existence of solutions in nonlinear shell theory. Izv. Akad. Nauk SSSR, Ser. Matem., Vol. 19, №4, 1956.
8. Krasnosel'skii, M. A., Topological Methods in the Theory of Nonlinear Integral Equations, (English translation). Pergamon Press, Book №10114, 1964.

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THEORY OF A TWO-DIMENSIONAL POTENTIAL FIELD IN PIECEWISE-NONHOMOGENEOUS ANISOTROPIC REGULAR MEDIA

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We consider a potential field in piecewise-nonhomogeneous media having a regular structure. The basic structure consists of a doubly-periodic system of groups of arbitrary heterogeneous anisotropic inclusions. The heterogeneous inclusions present in each of these groups possess the same periodicity as the basic structure; they thus form a substructure. The problem of uniquely determining the field in this structure reduces to a determination of the solutions of a second order homogeneous elliptic equation in each of the component domains, the solutions being required to satisfy coupling conditions on the interfaces of the media and also some additional relationships. This boundary-value problem reduces to a system of regular integral equations, which we prove to be solvable. Questions arise in connection with the modelling of piecewise-nonhomogeneous anisotropic regular structures of a general type by means of homogeneous anisotropic media. As applications, we consider certain problems in hydromechanics and in the theory of anisotropic reinforced materials.

1. Formulation of the basic problem. Let ω_1 and ω_2 $(\operatorname{Im}\omega_1 = 0,\ \operatorname{Im}\omega_2 / \omega_1 > 0)$ be the fundamental periods of the piecewise-nonhomogeneous medium, dividing it into a set of congruent fundamental cells Π_{mn} (for example, into a set of periodic parallelograms). Since we assume the structure of all congruent cells to be identical, it is sufficient to describe the structure of cell Π_{00}. The basic structure of the cell Π_{00} consists of a group of distinct heterogeneous anisotropic inclusions D_j, bounded by the closed curves L_j $(j = 1, 2, \ldots, r)$. The nonuniformity of each of the domains D_j gives rise to a cell substructure, i.e. the presence in each of these domains of its own anisotropic inclusions d_{jq}, bounded by the closed curves l_{jq} $(j = 1, 2, \ldots, r;\ q = 1, 2, \ldots, r_j)$. We assume that the curves L_j and l_{jq} are simple smooth mutually disjunct Liapunov curves.

Let

$$L = \bigcup_{j=1}^{r} L_j, \quad d_j = \bigcup_{q=1}^{r_j} d_{jq}, \quad l_j = \bigcup_{q=1}^{r_j} l_{jq}, \quad B_j = D_j \setminus d_j$$

and let D be the unbounded domain occupaied by the basic homogeneous anisotropic medium (see Fig. 1).

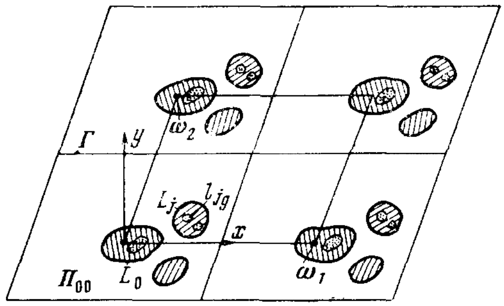

Fig. 1

Consider the scalar field in each of the domains D, B_j and d_{jq}, described therein by an equation of elliptic type

$$K_{11}(z)\,W_{xx}(z) + 2K_{12}(z)W_{xy}(z) + K_{22}(z)\,W_{yy}(z) = 0 \tag{1.1}$$

$$K_{\alpha\beta}(z) = \begin{cases} K_{\alpha\beta}, & z \in D \\ K^j_{\alpha\beta}, & z \in B_j, \\ K^{jq}_{\alpha\beta}, & z \in d_{jq} \end{cases} \qquad W(z) = \begin{cases} u(z), & z \in D \\ u_j(z), & z \in B_j \\ u_{jq}(z), & z \in d_{jq} \end{cases}$$

$$K_{11}(z)\,K_{22}(z) - K_{12}^2(z) > 0, \alpha,\ \beta = 1,2,\ K_{11}(z) > 0,\ K_{22}(z) > 0$$

Here $K_{\alpha\beta}$, $K^j_{\alpha\beta}$ and $K^{jq}_{\alpha\beta}$ are constants defining the physicotechnical properties of the anisotropic components of the structure. We define the flow q of the field at each point of the domain occupied by some component of the medium by the expression

$$q = q_1(z) + iq_2(z) = -[K_{11}(z)W_x - K_{12}(z)W_y] - i[K_{21}(z)\,W_x + K_{22}(z)W_y] \tag{1.2}$$

$$K_{12}(z) = K_{21}(z)$$

Here $q_\nu(z)$ assumes the values q_ν, q_ν^j and q_ν^{jq} in the domains D, B_j and d_{jq}, respectively.

We assume that the media occupying the congruent domains of the structure are identical in their physicotechnical properties; we assume also that for each $z \in D$ the following equations hold (q_n is the normal component of the vector q)

$$\int\limits_z^{z+\omega_\nu} q_n ds = -\sqrt{\Delta}\,\mathrm{Im}\,\Omega_\nu = \mathrm{const}, \quad \Delta = K_{11}K_{22} - K_{12}^2, \quad \nu = 1, 2 \tag{1.3}$$

Under these conditions the field in the unbounded piecewise-nonhomogeneous medium is completely determined by the field in the fundamental cell structure. Therefore we formulate the basic boundary-value problem in the following way.

In each of the domains D, B_j and d_{jp} we construct regular solutions of Eq. (1.1), satisfying the supplementary conditions (1.3) and the following boundary conditions on the interface of the medium components:

$$u(t) = u_j(t) + g_j(t), \quad q_n(t) = q_n^j(t), \qquad t \in L_j, \quad j = 1, 2, \ldots, r \tag{1.4}$$
$$u_j(t) = u_{jp}(t) + g_{jp}(t), \quad q_n^j(t) = q_n^{jp}(t), \qquad t \in l_{jp},$$
$$p = 1, 2, \ldots, r_j$$

The functions $g_j(t)$ and $g_{jp}(t)$ are arbitrary Hölder-continuous functions specified on L_j and l_{jp}, respectively.

We transform the boundary-value problem thus described to a more suitable form. With this in mind, we express the general solution of Eq. (1.1) in terms of arbitrary analytic functions. We have

$$u = \operatorname{Re}\varphi(z_0), \quad u_j = \operatorname{Re}\varphi_j(z_j), \quad u_{jq} = \operatorname{Re}\varphi_{jq}(z_{jq}) \tag{1.5}$$
$$z_0 = x + \mu_0 y, \quad z_j = x + \mu_j y, \quad z_{jq} = x + \mu_{jq} y$$
$$\mu(z) = \begin{cases} \mu_0 = \alpha_0 + i\beta_0, & z \in D \\ \mu_j = \alpha_j + i\beta_j, & z \in B_j, \\ \mu_{jq} = \alpha_{jq} + i\beta_{jq}, & z \in d_{jq}, \end{cases} \qquad \operatorname{Re}\mu(z) = -\frac{K_{12}(z)}{K_{22}(z)}, \quad |\mu(z)|^2 = \frac{K_{11}(z)}{K_{22}(z)}$$
$$\beta_0 > 0, \quad \beta_j > 0, \quad \beta_{jq} > 0, \quad j = 1, 2, \ldots, r; \quad q = 1, 2, \ldots, r_j$$

The periodicity of the structure is maintained in the z_0 plane, wherein the fundamental periods ω_{10} and ω_{20} now have the form

$$\omega_{10} = \omega_1, \quad \omega_{20} = \operatorname{Re}\omega_2 + \mu_0 \operatorname{Im}\omega_2 = \tag{1.6}$$
$$h + \alpha_0 H + i\beta_0 H = h_0 + iH_0$$
$$h = \operatorname{Re}\omega_2, \ H = \operatorname{Im}\ \omega_2, \quad h_0 = \operatorname{Re}\omega_{20}, \ H_0 = \operatorname{Im}\ \omega_{20}$$

Given pre-images in the z plane, consisting of points, curves, and regions, have corresponding images under an affine mapping in the z_0, z_j and z_{jq} ($j = 1, 2, \ldots, r$; $q = 1, 2, \ldots, r_j$) planes; pre-images in the z plane corresponding to images in these latter planes will be identified by zero, prime, and double-prime superscripts, respectively.

We now calculate the flow Q crossing an arbitrary curve joining points A and B in D. Taking the relations (1.5) and (1.2) into account, we obtain

$$Q = \int_A^B q_n ds = \int_A^B (q_1 dy - q_2 dx) = -\sqrt{\Delta}\operatorname{Im}\varphi(z_0)\Big|_{A_0}^{B_0} \tag{1.7}$$

This latter relation then enables us to write the supplementary relations (1.3) in the form

$$\operatorname{Im}\{\varphi(z_0 + \omega_{10}) - \varphi(z_0)\} = \operatorname{Im}\ \Omega_1 \tag{1.8}$$
$$\operatorname{Im}\{\varphi(z_0 + \omega_{20}) - \varphi(z_0)\} = \operatorname{Im}\ \Omega_2$$

From Eqs. (1.8) it follows that the function $\varphi(z_0)$, regular in D°, is quasi-periodic with respect to the periods ω_{10} and ω_{20}. The boundary conditions (1.4) can be written in the following equivalent form when (1.7) is taken into account:

$$\varphi(t_0) = \varepsilon_j \varphi_j(t_j) + \varepsilon_j^* \overline{\varphi_j(t_j)} + g_j(t) \tag{1.9}$$
$$t \in L_j, \qquad t_0 \in L_j^\circ, \quad t_j \in L_j'$$
$$\varphi_j(t_j) = \varepsilon_{jq}\varphi_{jq}(t_{jq}) + \varepsilon_{jq}^* \overline{\varphi_{jq}(t_{jq})} + g_{jq}(t)$$
$$t \in l_{jq}, \quad t_j \in l_{jq}', \qquad t_{jq} \in l_{jq}''$$

$$\varepsilon_j = \frac{1}{2}(1+\lambda_j),\quad \varepsilon_j^* = \frac{1}{2}(1-\lambda_j),\quad \varepsilon_{jq} = \frac{1}{2}(1+\lambda_{jq}),\quad \varepsilon_{jq}^* = \frac{1}{2}(1-\lambda_{jq})$$

$$\lambda_j = \frac{\beta_j K_{22}{}^j}{\beta K_{22}},\quad \lambda_{jq} = \frac{\beta_{jq} K_{22}{}^{jq}}{\beta_j K_{22}{}^j},\quad j=1,2,\ldots,r,\quad q=1,2,\ldots,r_j$$

The constants of integration, which must appear in the right-hand sides of (1.9), are included in the unknown functions φ_j and φ_{jq}. We have thus arrived at the following boundary-value problem.

Determine a function $\varphi(z_0)$, quasi-periodic in D°, and functions $\varphi_j(z_j)$ and $\varphi_{jq}(z_{jq})$, regular in the domains B_j' and d_{jq}'', respectively, which satisfy the boundary conditions (1.9) and the supplementary conditions (1.8). It is, of course, to be understood that the conditions for $\varphi(z_0)$ to be quasi-periodic are satisfied automatically owing to the special representation chosen for the function $\varphi(z_0)$.

We set

$$\varphi(z_0) = \frac{1}{2\pi i}\int_{L^\bullet} \{\varepsilon(t)P(t) - \varepsilon^*(t)\overline{P(t)}\}\zeta(t_0 - z_0)\,dt_0 + Az_0,\qquad z_0 \in D \tag{1.10}$$

$$\varphi_j(z_j) = \frac{1}{2\pi i}\int_{L_j'} \frac{P_j(t)}{t_j - z_j}\,dt_j + \frac{1}{2\pi i}\sum_{p=1}^{r_j}\int_{l_{jp}'}\{\varepsilon_{jp}P_{jp}(t) - \varepsilon_{jp}^*\overline{P_{jp}(t)}\}\frac{dt_j}{t_j - z_j},\quad z_j \in B_j'$$

$$\varphi_{jq}(z_{jq}) = \frac{1}{2\pi i}\int_{l_{jq}''}\frac{P_{jq}(t)}{t_{jq} - z_{jq}}\,dt_{jq},\quad z_{jq} \in d_{jq}'',\quad j=1,2,\ldots,r,\quad q=1,2,\ldots,r_j$$

$$t_0 = \frac{t}{2}(1-i\mu_0) + \frac{\bar t}{2}(1+i\mu_0),\quad z_0 = \frac{z}{2}(1-i\mu_0) + \frac{\bar z}{2}(1+i\mu_0)$$
$$t \in L,\qquad z \in D$$

$$t_j = \frac{t}{2}(1-i\mu_j) + \frac{\bar t}{2}(1+i\mu_j),\quad z_j = \frac{z}{2}(1-i\mu_j) + \frac{\bar z}{2}(1+i\mu_j)$$
$$t \in L_j + l_j,\qquad z \in B_j$$

$$t_{jq} = \frac{t}{2}(1-i\mu_{jq}) + \frac{\bar t}{2}(1+i\mu_{jq}),\ z_{jq} = \frac{z}{2}(1-i\mu_{jq}) + \frac{\bar z}{2}(1+i\mu_{jq})$$
$$t \in l_{jq},\qquad z \in d_{jq}$$

$$t_0 \in L^\circ,\quad t_j \in L_j' + l_j',\quad t_{jq} \in l_{jq}''$$

$$P(t) = \{P_j(t),\ t \in L_j\},\quad \varepsilon(t) = \{\varepsilon_j\, t \in L_j\},\quad \varepsilon^*(t) = \{\varepsilon_j^*,\ t \in L_j\}$$

Here $\zeta(z_0)$ is the Weierstrass zeta function constructed on the periods ω_{10} and ω_{20}, and A is a constant to be defined. The curves l_{jq}'' and l_{jp}' are traversed clockwise in the integration while the curves L_j' and L_j° are traversed counterclockwise. The function $\varphi(z_0)$, defined in (1.10), is obviously quasi-periodic in D°.

Substituting the increment of $\varphi(z_0)$ into (1.8) and solving the resulting equation for A, we obtain

$$A = A_L + A_\Omega \tag{1.11}$$

$$A_L = \left(\frac{\delta_{10}}{\omega_{10}} - \frac{\pi}{S_0}\right)a - \frac{\pi}{S_0}\bar a,\quad A_\Omega = \frac{1}{S_0}(\bar\omega_{10}\,\mathrm{Im}\,\Omega_2 - \bar\omega_{20}\,\mathrm{Im}\,\Omega_1)$$

$$\delta_{10} = \zeta(z_0 + \omega_{10}) - \zeta(z_0) = 2\zeta\left(\frac{\omega_{10}}{2}\right)$$

$$\delta_{20} = \zeta(z_0 + \omega_{20}) - \zeta(z_0) = 2\zeta\left(\frac{\omega_{20}}{2}\right)$$

$$a = \frac{1}{2\pi i}\int\limits_{L^\circ} \{\varepsilon(t)\,P(t) - \varepsilon^*(t)\,\overline{P(t)}\}\,dt_0, \qquad S_0 = \omega_{10}\,\mathrm{Im}\,\omega_{20}$$

Thus the formulation given by (1.10) defines a function $\varphi\ (z_0)$, which is quasi-periodic in D° and satisfies the supplementary conditions (1.8). The problem now reduces to determining the densities $P_j\ (t)$ and $P_{jq}\ (t)$ from the boundary conditions (1.9).

2. Solution of the boundary-value problem (1.9). If we pass over to the limiting values in Eqs. (1.10) and substitute them into the boundary conditions (1.9), we obtain a system of Fredholm integral equations of the second kind in the unknown functions P_j and P_{jq}:

$$P_j(\tau) - M_j\{P_j(t),\ P_{jq}(t),\ \tau\} = F_j(\tau), \tau \in L_j, \qquad j = 1, 2, \ldots, r \quad (2.1)$$

$$P_{jq}(\tau) - M_{jq}\{P_{jq}(t), P_j(t), \tau\} = F_{iq}(\tau), \qquad \tau \in l_{jq}, \qquad q = 1, 2, \ldots, r_j$$

$$M_j = \frac{\varepsilon_j^*}{2\pi i\varepsilon_j}\int\limits_{L_j}\overline{P_j(t)}\,d\left\{\ln\frac{\bar t_j - \bar\tau_j}{\sigma(t_0 - \tau_0)}\right\} - \frac{1}{2\pi i}\int\limits_{L_j} P_j(t)\,d\left\{\ln\frac{t_j - \tau_j}{\sigma(t_0 - \tau_0)}\right\} +$$

$$\frac{1}{2\pi i\varepsilon_j}\int\limits_{L^\circ\setminus L^\circ_j}\{\varepsilon(t)\,P(t) - \varepsilon^*(t)\,\overline{P(t)}\}\,\zeta(t_0 - \tau_0)\,dt_0 +$$

$$\frac{\varepsilon_j^*}{2\pi i\varepsilon_j}\sum_{q=1}^{r_j}\int\limits_{l_{jq}'}\{\varepsilon_{jq}\overline{P_{jq}(t)} - \varepsilon_{jq}^*(tP_{jq})\}\frac{d\bar t_j}{\bar t_j - \bar\tau_j} -$$

$$\frac{1}{2\pi i}\sum_{q=1}^{r_j}\int\limits_{l_{jq}'}\{\varepsilon_{jq}P_{jq}(t) - \varepsilon_{jq}^*\,\overline{P_{jq}(t)}\}\frac{dt_j}{t_j - \tau_j} + \frac{\tau_0}{\varepsilon_j}A_L$$

$$M_{jq} = \frac{1}{2\pi i\varepsilon_{jq}}\sum_{s=1}^{r_j}{}'\int\limits_{l_{js}'}\{\varepsilon_{js}^*\overline{P_{js}(t)} - \varepsilon_{js}P_{js}(t)\}\frac{dt_j}{t_j - \tau_j} -$$

$$\frac{1}{2\pi i}\int\limits_{l_{jq}} P_{iq}(t)\,d\left\{\ln\frac{t_j - \tau_j}{t_{jq} - \tau_{jq}}\right\} - \frac{\varepsilon_{jq}^*}{2\pi i\varepsilon_{jq}}\int\limits_{l_{jq}}\overline{P_{jq}(t)}\,d\left\{\ln\frac{\bar t_{jq} - \bar\tau_{jq}}{t_j - \tau_j}\right\} -$$

$$\frac{1}{2\pi i\varepsilon_{jq}}\int\limits_{L_j'}\frac{P_j(t)}{t_j - \tau_j}\,dt_j$$

$$F_j(\tau) = \frac{1}{\varepsilon_j}\left[\tau_0 A_\Omega - g_j(\tau)\right], \qquad F_{jq}(\tau) = \frac{1}{\varepsilon_{jq}}g_{jq}(\tau)$$

The prime on the summation in the expression for M_{jq} means that the term corresonding to $s = q$ must be omitted. If the system (2.1) is solvable, its solution yields the unknown functions φ, φ_j and φ_{jq}.

3. Uniqueness theorems. Let us assume that solutions of the boundary-value problem (1.9), (1.8) exist.

Theorem 3.1. The following relationships hold between any two solutions of the boundary-value problem (1.9) $\varphi^{(1)}(z_0)$, $\varphi_j^{(1)}(z_j)$, $\varphi_{jq}^{(1)}(z_{jq})$ and $\varphi^{(2)}(z_0)$, $\varphi_j^{(2)}(z_j)$, $\varphi_{jq}^{(2)}(z_{jq})$, each of which satisfies the supplementary conditions (1.8):

$$\varphi^\circ = \varphi^{(2)}(z_0) - \varphi^{(1)}(z_0) = C, \quad \varphi_j^\circ = \varphi_j^{(2)}(z_j) - \varphi_j^{(1)}(z_j) = C_j \tag{3.1}$$

$$\overset{\circ}{\varphi}_{jq} = \varphi_{jq}^{(2)}(z_{jq}) - \varphi_{jq}^{(1)}(z_{jq}) = C_{jq}, \qquad j = 1, 2, \ldots, r; \quad q = 1, 2, \ldots, r_j$$

$$C_j = \operatorname{Re} C + \frac{i}{\lambda_j} \operatorname{Im} C, \qquad C_{jq} = \operatorname{Re} C + \frac{i}{\lambda_j \lambda_{jq}} \operatorname{Im} C$$

To prove this theorem we note, first of all, that the "energy" equality holds for any arbitrary solution of Eq. (1.1) regular in the domain B , namely,

$$\iint_B |W_y - \mu W_x|^2 \, dx\, dy = -\frac{1}{K_{22}} \int_{L_B} W q_n \, ds \tag{3.2}$$

Here L_B is the boundary of the domain B, q_n is the normal component of the flow q, introduced in (1.2); the integration is taken in a counterclockwise direction.

The Eq. (3.2) is derived by the usual transformations, analogous to those which lead to the Poisson's integral formula of potential theory. In fact, when $K_{11} = K_{22}$, $K_{12} = 0$, this equation becomes Poisson's formula.

If now we apply (3.2) to our multi-component structure, upon taking into account (1.1), (1.9) and (1.5), we obtain

$$J = K_{22} \iint_{D_\Gamma} \left| \frac{\partial u}{\partial y} - \mu_0 \frac{\partial u}{\partial x} \right|^2 dx\, dy + \tag{3.3}$$

$$\sum_{j=1}^{r} K_{22}{}^j \iint_{B_j} \left| \frac{\partial u_j}{\partial y} - \mu_j \frac{\partial u_j}{\partial x} \right|^2 dx\, dy +$$

$$\sum_{j=1}^{r} \sum_{q=1}^{r_j} K_{22}^{jq} \iint_{d_{jq}} \left| \frac{\partial u_{jq}}{\partial y} - \mu_{jq} \frac{\partial u_{jq}}{\partial x} \right|^2 dx\, dy =$$

$$\sum_{j=1}^{r} \sum_{q=1}^{r_j} \int_{l_{jq}} q_n{}^j g_{jq}(t)\, ds + \sum_{j=1}^{r} \int_{L_j} q_n g_j(t)\, ds - \int_\Gamma u q_n \, ds$$

In (3.3) the contours l_{jq}, L_j and Γ (boundary of Π_{00}) are traversed counterclockwise; D_Γ is an ($r + 1$)-connected domain with the boundary $\Gamma \bigcup L$.

Taking note of the relations (1.5), (1.7) and (1.8), by virtue of quasi-periodic property of $\varphi(z_0)$, we obtain

$$\int_\Gamma u q_n \, ds = (\operatorname{Re} \Omega_2 \operatorname{Im} \Omega_1 - \operatorname{Re} \Omega_1 \operatorname{Im} \Omega_2) \sqrt{\Delta} \tag{3.4}$$

If we substitute Eq. (3.4) into the energy equality (3.2) and then apply relation (3.3) to the difference of two solutions of the boundary-value problem (1.9), wherein these solutions satisfy the condition (1.8), we obtain the result asserted in the theorem. The functions φ°, $\varphi_j{}^\circ$ and $\varphi_{jq}{}^\circ$ can be interpreted as the solutions of the homogeneous boundary-value problem (1.9) corresponding to

$$g_j(t) \equiv 0, \; g_{jq}(t) \equiv 0, \qquad \operatorname{Im} \Omega_1 = \operatorname{Im} \Omega_2 = 0$$

T h e o r e m 3.2. Let $\chi_j(t)$ and $\sigma_j(t)$ be the boundary values of the functions $\chi_j(z_0)$ and $\sigma_j(z_j)$, regular, respectively, in a finite domain of the z_0 plane bounded by the curve $L_j^\bullet$ and in the complement of D_j' taken with respect to the extended z_j plane. If $\sigma_j(z_j) = O(|z_j|^{-1})$, in the neighborhood of the point at infinity, then the boundary-value problem

$$\chi_j(t) = \varepsilon_j\sigma_j(t) + \varepsilon_j^*\overline{\sigma_j(t)}, \quad t \in L_j, \quad j = 1, 2, \ldots, r$$

has only the trivial solution.

To prove this theorem, we apply (3.2) to the two-component domain bounded by a circle C_R of sufficiently large radius. We have

$$\beta_0^2 K_{22} \iint\limits_{D_j} |\chi'(z_0)|^2\, dx\, dy + \beta_j^2 K_{22}{}^j \iint\limits_{D_R} |\sigma_j'(z_j)|^2\, dx\, dy = \int\limits_{C_R} q_n \operatorname{Re}\sigma_j(z_j)\, ds, \qquad \chi' = \frac{d\chi}{dz_0} \tag{3.5}$$

Here D_R is the doubly-connected domain with boundary $L_j \bigcup C_R$. We then obtain the desired result by letting R become infinite.

4. Solvability of the system (2.1). We prove now that the system of integral equations (2.1) is always solvable under the assumptions we have made on the boundary curves and the functions g_j and g_{jq}. To do this we consider the corresponding homogeneous system. It is obvious tnat $F_j(\tau) = 0$ and $F_{jq}(\tau) = 0$ if and only if

$$g_j(\tau) = 0, \; g_{jq}(\tau) = 0, \qquad \operatorname{Im}\Omega_1 = \operatorname{Im}\Omega_2 = 0 \tag{4.1}$$
$$j = 1, 2, \ldots, r; \qquad q = 1, 2, \ldots, r_j$$

Thus the homogeneous system corresponds to the homogeneous boundary-value problem (1.9) with homogeneous supplementary conditions (1.8).

We denote the solutions of the homogeneous system (2.1) by $P_j^\circ(t)$ and $P_{jq}^\circ(t)$. Functions and functionals corresponding to these solutions will also be labelled with a zero sub- or superscript. The uniqueness theorem (3.1) enables us to write

$$\varphi^\circ(z_0) = \frac{1}{2\pi i}\int\limits_{L^\circ} \{\varepsilon(t)P_0(t) - \varepsilon^*(t)\overline{P_0(t)}\}\,\zeta(t_0 - z_0)\,dt_0 + A_0 z_0 = C, \quad z_0 \in D^\circ \tag{4.2}$$

$$\varphi_j^\circ(z_j) = \frac{1}{2\pi i}\int\limits_{L_j'} \frac{P_j^\circ(t)}{t_j - z_j}\,dt_j + \frac{1}{2\pi i}\sum_{s=1}^{r_j}\int\limits_{l_{js}'} \{\varepsilon_{js}P_{js}^\circ(t) - \varepsilon_{js}^*\overline{P_{js}^\circ(t)}\}\frac{dt_j}{t_j - z_j} = C_j, \quad z_j \in B_j'$$

$$\varphi_{jq}^\circ(z_{jq}) = \frac{1}{2\pi i}\int\limits_{l_{jq}''} \frac{P_{jq}^\circ(t)}{t_{jq} - z_{jq}}\,dt_{jq} = C_{jq}, \qquad z_{jq} \in d_{jq}''$$

Calculating the increments of the function $\varphi^\circ(z_0)$ in the first of the equations (4.2) for a passage from the point z to its congruent point $z + \omega_\nu$ $(\nu = 1,2)$, we obtain

$$\frac{1}{2\pi i}\int\limits_{L^\circ} \{\varepsilon(t)P_0(t) - \varepsilon^*(t)\overline{P_0(t)}\}\,\zeta(t_0 - z_0)\,dt_0 = C \tag{4.3}$$
$$A_0 = a_0 = 0, \qquad z_0 \in D^\circ$$

It follows from (4.3) that the function $\varepsilon(t)P_0(t) - \varepsilon^*(t)\overline{P_0(t)}$ is the boundary value of certain functions regular in finite domains of the z_0 plane bounded by the contours L_j° $(j = 1, 2, \ldots, r)$. Therefore the integral in Eq. (4.3) vanishes and, on the basis of (3.1), we obtain

$$C = 0, \quad C_j = 0, \quad C_{jq} = 0, \qquad j = 1, 2, \ldots, r; \; q = 1, 2, \ldots, r_j \tag{4.4}$$

We now introduce the functions

$$i\chi_j(t) = \varepsilon_j P_j^\circ(t) - \varepsilon_j^* \overline{P_j^\circ(t)}, \quad i\sigma_j(t) = P_j^\circ(t), \quad t \in L_j, \quad j = 1, 2, \ldots, r \tag{4.5}$$
$$i\chi_{jq}(t) = \varepsilon_{jq} P_{jq}^\circ(t) - \varepsilon_{jq}^* \overline{P_{jq}^\circ(t)}, \quad i\sigma_{jq}(t) = P_{jq}^\circ(t)$$
$$t \in l_{jq}, \qquad q = 1, 2, \ldots, r_j$$

From (4.2), taking account of (4.4) and (4.5), we conclude that $\chi_j(t)$ is the boundary value of the functions $\chi_j(z_0)$, which are regular in the finite domains bounded by the curves L_j°; $\sigma_j(t)$ is the boundary value of the functions $\sigma_j(z_j)$, which are regular outside of D_j' and vanish at infinity; $\chi_{jq}(t)$ is the boundary value of the functions $\chi_{jq}(z_j)$, which are regular in the finite domains bounded by the curves l_{jq}', and, finally, $\sigma_{jq}(t)$ is the boundary value of the functions $\sigma_{jq}(z_{jq})$, which are regular outside of the domains d_{jq}'' and vanish at infinity.

If we eliminate the functions $P_j^\circ(t)$ and $P_{jq}^\circ(t)$, from (4.5), we obtain the system of independent boundary-value problems

$$\chi_j(t) = \varepsilon_j \sigma_j(t) + \varepsilon_j^* \overline{\sigma_j(t)}, \quad t \in L_j, \quad j = 1, 2, \ldots, r \tag{4.6}$$
$$\chi_{jq}(t) = \varepsilon_{jq}\sigma_{jq}(t) + \varepsilon_{jq}^* \overline{\sigma_{jq}(t)}, \quad t \in l_{jq}, \quad q = 1, 2, \ldots, r_j$$

In the neighborhood of the point at infinity the functions $\sigma_j(z_j)$ and $\sigma_{jq}(z_{jq})$ decay no slower than $|z_j^{-1}|$ and $|z_{jq}^{-1}|$, respectively. By virtue of Theorem 3.2 we have

$$\sigma_j(t) = \chi_j(t) = 0, \quad \sigma_{jq}(t) = \chi_{jq}(t) = 0, \quad j = 1, 2, \ldots, r; q = 1, 2, \ldots, r_j \tag{4.7}$$

Hence on the basis of (4.5) we conclude that

$$P_j^\circ(t) = 0, \quad P_{jq}^\circ(t) = 0, \quad j = 1, 2, \ldots, r; \quad q = 1, 2, \ldots, r_j \tag{4.8}$$

Thus the system of equations (2.1) always has a solution and this solution is unique.

It is evident from the proof that, without making any major changes, we can complicate the structure of the fundamental cell even further by introducing substructures of much higher orders. By letting the periods become infinite, we obtain the solution of the boundary-value problem (4.5) for a multiply connected domain with a finite number of components.

5. Model of a regular field (of regular structure). Let us set $g_j(t) = 0$, $g_{jq}(t) = 0$.

In each component of the medium there is a scalar field defined by the functions u, u_j and u_{jq}. The flow at each field point is given by expression (1.2), and the total flow across an arc joining two congruent points is independent of z and is determined in (1.3).

We introduce mean flows $\langle q_1 \rangle$ and $\langle q_2 \rangle$ according to the formulas

$$\omega_1 \langle q_2 \rangle = \int_{z+\omega_1}^{z} q_n ds = \sqrt{\Delta}\, \mathrm{Im}\, \Omega_1, \qquad h\langle q_2 \rangle - H\langle q_1 \rangle = \sqrt{\Delta}\, \mathrm{Im}\, \Omega_2 \tag{5.1}$$

The mean gradients $\langle u_x \rangle$ and $\langle u_y \rangle$ in the structure have the form

$$\begin{aligned} \langle u_x \rangle \omega_1 &= \mathrm{Re}\,\varphi(z_0 + \omega_{10}) - \mathrm{Re}\,\varphi(z_0) = \mathrm{Re}\,\Omega_1 \\ \langle u_x \rangle h + \langle u_y \rangle H &= \mathrm{Re}\,\varphi(z_0 + \omega_{20}) - \mathrm{Re}\,\varphi(z_0) = \mathrm{Re}\,\Omega_2 \end{aligned} \tag{5.2}$$

We find expressions for $\mathrm{Re}\,\Omega_1$ and $\mathrm{Re}\,\Omega_2$ in terms of $\mathrm{Im}\,\Omega_1$ and $\mathrm{Im}\,\Omega_2$, wherein we take into account (1.10) and (1.11); thus,

$$\begin{aligned} \mathrm{Re}\,\Omega_1 &= \frac{\omega_{10}}{H_0}\mathrm{Im}\,\Omega_2 - \frac{h_0}{H_0}\mathrm{Im}\,\Omega_1 - \frac{2\pi}{H_0}\mathrm{Re}\,a \\ \mathrm{Re}\,\Omega_2 &= \frac{h_0}{H_0}\mathrm{Im}\,\Omega_2 - \frac{|\omega_{20}|^2}{S_0}\mathrm{Im}\,\Omega_1 - \frac{2\pi}{\omega_{10}}\mathrm{Im}\,a - \frac{2\pi h_0}{S_0}\mathrm{Re}\,a \end{aligned} \tag{5.3}$$

We introduce a functional a, written in the form

$$a = a_1\,\mathrm{Im}\,\Omega_1 + a_2\,\mathrm{Im}\,\Omega_2 \tag{5.4}$$

Here a_1 is the functional a corresponding to the solution $P\ (t)$ with $\mathrm{Im}\ \Omega_1 = 1$ and $\mathrm{Im}\ \Omega_2 = 0$, and a_2 is the functional a corresponding to $P(t)$ for $\mathrm{Im}\ \Omega_2 = 1$ and $\mathrm{Im}\ \Omega_1 = 0$.

If now we substitute the expressions (5.3) into the right-hand sides of Eqs. (5.2) and take into account the relations (5.1) and (5.4), we obtain the law relating mean flows and mean gradients in the structure, namely ,

$$\begin{aligned} &\langle u_x \rangle = \varkappa_{11}\langle q_1 \rangle + \varkappa_{12}\langle q_2 \rangle, \qquad \langle u_y \rangle = \varkappa_{21}\langle q_1 \rangle + \varkappa_{22}\langle q_2 \rangle \\ &\varkappa_{11} = \frac{K_{22}}{\Delta}\left(\frac{2\pi}{\omega_1}\mathrm{Re}\,a_2 - 1\right), \qquad \Delta = K_{11}K_{22} - K_{12}^2 \\ &\varkappa_{12} = \frac{1}{\Delta}\left[K_{12} - \frac{2\pi}{H}K_{22}\,\mathrm{Re}\left(a_1 + \frac{h}{\omega_1}a_2\right)\right] \\ &\varkappa_{21} = \frac{1}{\Delta}\left\{K_{12} + \frac{2\pi}{\omega_1}\mathrm{Im}\left[a_2\left(\sqrt{\Delta} - i\frac{K_{12}}{H_0}\right)\right]\right\} \\ &\varkappa_{22} = -\frac{1}{\Delta}\left\{K_{11} + \frac{2\pi}{H}\mathrm{Im}\left[\left(a_1 + \frac{h}{\omega_1}a_2\right)\left(\sqrt{\Delta} - i\frac{K_{12}}{H_0}\right)\right]\right\} \end{aligned} \tag{5.5}$$

When $a_1 = a_2 = 0$ (a homogeneous anisotropic medium), the law (5.5) reduces to that given in (1.2).

We call the coefficients of $\langle q_i \rangle$ in the law (5.5) macroscopic parameters of the structure and we refer to an homogeneous anisotropic medium with these parameters as a model medium.

Theorem 5.1. The macroscopic parameters of a structure form a symmetric non-singular matrix $\varkappa = \|\varkappa_{ik}\|$.

To prove this theorem we replace $\mathrm{Re}\ \Omega_\nu$ and $\mathrm{Im}\ \Omega_\nu\ (\nu = 1,2)$ in the right member of Eq. (3.4) by their expressions from (5.1) and (5.2). Equation (3.3) can then be written in the form

$$J = -S\{\langle q_1 \rangle\langle u_x \rangle + \langle q_2 \rangle\langle u_y \rangle\} \tag{5.6}$$

Introducing the standard solutions $u^{(\nu)}$, $u_j^{(\nu)}$ and $u_{jq}^{(\nu)}$ $(\nu = 1,2)$ in accord with the formulas

$$\begin{aligned} u &= u^{(1)}\langle q_1 \rangle + u^{(2)}\langle q_2 \rangle \\ u_j &= u_j^{(1)}\langle q_1 \rangle + u_j^{(2)}\langle q_2 \rangle \\ u_{jq} &= u_{jq}^{(1)}\langle q_1 \rangle + u_{jq}^{(2)}\langle q_2 \rangle \end{aligned} \tag{5.7}$$

and then substituting the latter into the left side of Eq. (3.3), we obtain (the meaning of J_{ik} is clear from the text)

$$\sum_{i,k=1}^{2} J_{ik}\langle q_i\rangle\langle q_k\rangle = -S\{\langle q_1\rangle\langle u_x\rangle + \langle q_2\rangle\langle u_y\rangle\}, \quad J_{ik} = J_{ki} \qquad (5.8)$$

If we now differentiate (5.8) with respect to $\langle q_\nu\rangle$ ($\nu = 1,2$), we obtain the law (5.5). Its coefficient matrix $\varkappa$ is obviously symmetric. Since the quadratic form on the left side of Eq. (5.8) is positive definite, we have $\det \varkappa \neq 0$.

The following theorem summarizes the results of Sect. 5.

Theorem 5.2. For a regular piecewise-nonhomogeneous anisotropic structure with quasi-periodic flows Q there exists a model homogeneous anisotropic medium governed by the law (5.5).

6. Applications [1, 2].

Potential flows in anisotropic porous media. Consider a plane-parallel potential flow of a liquid in a piecewise-homogeneous anisotropic porous medium whose cross section is a regular domain of the type of structure considered in Sect. 1. Upon requiring that the flow rate of the liquid across an arbitrary curve joining two congruent points z and $z+\omega_\nu$ ($\nu=1, 2$) of the medium be constant and be equal, respectively, to $-\sqrt{\Delta}\,\mathrm{Im}\,\Omega_1$ and $-\sqrt{\Delta}\,\mathrm{Im}\,\Omega_2$, we obtain the boundary-value problem (1.4) subject to the supplementary conditions (1.3). We interpret $K_{\alpha\beta}$, $K^{j}_{\alpha\beta}$, $K^{jq}_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) as the filtration coefficients in the anisotropic components of the porous medium (relation (1.2) is Darcy's law for each component of the medium). We interpret u, u_j and u_{jq} as the liquid pressure in the medium components D, B_j and d_{jq}, respectively. The unknown functions (velocities and pressure in the liquid) are completely described by the relations (1.5), (1.10), and by the system of integral equations (2.1). Theorem 5.2 assures us that for the piecewise-nonhomogeneous porous anisotropic medium in question there exists a model homogenous porous anisotropic medium governed by the Darcy law (5.5).

Transverse thermal conductivity of reinforced anisotropic media. Consider an anisotropic medium reinforced by congruent groups of heterogeneous anisotropic fibers (which can, in turn, be reinforced by heterogeneous fibers so that the cross section of such a medium is the type of structure described in Sect. 1) and suppose that the medium is permeated by a stationary thermal flow normal to the fiber axes with the mean cell values $\langle q_1\rangle$ and $\langle q_2\rangle$. In this case we again have the boundary-value problem (1.4), (1.3) for the temperature u, u_j and u_{jq} in each medium component. The quantities $K_{\alpha\beta}$, $K^{j}_{\alpha\beta}$, $K^{jq}_{\alpha\beta}$ ($\alpha, \beta = 1,2$) are now to be interpreted as thermal conductivity coefficients of the corresponding components of the medium, and the relations (1.2) are to be interpreted as the law governing the thermal conduction in each component. According to Theorem 5.2 the reinforced medium can be replaced by a homogeneous model medium governed by the law (5.5).

Electrostatic field in anisotropic reinforced dielectrics. If an unbounded anisotropic piecewise-nonhomogeneous medium with a cross section of the type considered in Sect. 1 is penetrated by a transverse electrical field with the same mean vector intensity $\langle E\rangle = \langle q_1\rangle - i\langle q_2\rangle$ in each fundamental cell, we again have the boundary-value problem (1.4), (1.3) for the field potential u, u_j, u_{jq} in the corresponding media components. The quantities $K_{\alpha\beta}$, $K^{j}_{\alpha\beta}$, $K^{jq}_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are then the dielectric permeabilities of the material in the corresponding components of the medium. The model homogeneous anisotropic medium corresponding to the structure is governed by the law (5.5), where K_{11}, K_{12}, K_{22} are the dielectric permeabilities of the matrix material. If the coupling media (or only some of them) are isotropic, then all our results remain valid. We need only set $K_{12} = 0$, $K_{11} = K_{22} = K$ for each corresponding domain.

REFERENCES

1. Landau, L. D. and Lifshits, E. M., Fluid Mechanics. Addison-Wesley, Reading, Mass., 1959.
2. Polubarinova-Kochina, P. Ia. and Fal'kovich, S. V., Theory of filtration of liquids in porous media. PMM Vol. 11, №6, 1947.

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CORRELATION FUNCTIONS OF THE ELASTIC FIELD OF MULTIPHASE POLYCRYSTALS

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An approximation of the homogeneity of a linear combination of the stresses and strains $\sigma + b\varepsilon = \text{const}$ is proposed to evaluate the correlation functions of the elastic field of micro-inhomogeneous media. This approximation is a generalisation of the Voigt and Reuss hypotheses according to which the strains ε and the stresses σ are considered homogeneous, respectively. Independence of the spatial fluctuations of the volume and shear components of the elastic field holds within the scope of the approximation made. It is shown that the proposed relationship is satisfied exactly for laminar materials, but approximately for fibrous and granular materials. An explicit form is found for the tensor $\mathbf{b}$ in the singular approximation of random function theory under the assumption of isotropy of the properties of each of the fibrous and granular material phases and the correlation functions and stress and strain fields dispersions are calculated. It is shown that in this approximation the coordinate and tensor dependences of the correlation functions of the stress and strain fields are separated. An analogous computation is performed for multiphase polycrystals in the correlation approximation according to which correlation functions of elastic moduli of not higher than the second order are taken into account. In this approximation, the coordinate and tensor dependences of the correlation functions of the elastic field do not separate. Conditions are found under which the correlation approximation results in independence of the volume and shear components of the elastic field fluctuations.

The exact computation of the stress and strain fields is a complex problem for the deformation of micro-inhomogeneous media (composite materials, single-